MATH 504 HOMEWORK 9

Due Friday, December 7.
This homework is worth two regular homework sets.

Problem 1. *Let T be a normal Suslin tree, and let $(\mathbb{P}_T, <) = (T, >)$. Show that although $\mathbb{P}_T$ has the countable chain condition, $\mathbb{P}_T \times \mathbb{P}_T$ does not. Hint: for every $x \in T$, pick two immediate successors p_x, q_x of x. Look at the set $\{(p_x, q_x) \mid x \in T\} \subset \mathbb{P}_T \times \mathbb{P}_T$.*

Problem 2. *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are two c.c.c. posets. Show that the following are equivalent:*

(1) $\mathbb{P} \times \mathbb{Q}$ *is c.c.c;*
(2) $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \check{\mathbb{Q}}$ *is c.c.c;*
(3) $1_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \check{\mathbb{P}}$ *is c.c.c;*

Problem 3. *Let $\mathbb{P}$ be a poset such that for every $p \in \mathbb{P}$, there are incompatible $q, r \leq p$. Suppose G is $\mathbb{P}$-generic. Show that $G \times G$ is not $\mathbb{P} \times \mathbb{P}$-generic.*

Problem 4. *Let $\mathbb{P} \in V$ be a poset, and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$ name for a poset, i.e. $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is a poset. Suppose that G is $\mathbb{P}$-generic over V, and that H is $\dot{\mathbb{Q}}_G$-generic over $V[G]$. Show that $K := G * H = \{(p, \dot{q}) \mid p \in G, \dot{q}_G \in H\}$ is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V.*

Problem 5. *Suppose that $\mathbb{P} * \dot{\mathbb{Q}}$ has the κ-chain condition. Show that $\mathbb{P}$ has the κ-chain condition, and $1_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{Q}}$ has the κ-chain condition".*

Remark 1. The converse is also true.

Problem 6. *Suppose that $\mathbb{P}$ is κ distributive, and $1_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{Q}}$ is κ-distributive". Show that that $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-distributive.*

Problem 7. *Suppose that $\mathbb{P} = \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta \mid \beta < \alpha \rangle$ is an iteration of length α, and $\mathbb{P}_\beta = \mathbb{P}_\alpha \restriction \beta$ for $\beta < \alpha$. Show that if G_α is $\mathbb{P}_\alpha$-generic and we define $G_\beta := \{p \restriction \beta \mid p \in G_\alpha\}$, then G_β is $\mathbb{P}_\beta$-generic and $V[G_\beta] \subset V[G_\alpha]$. (Note that we are not assuming finite support here, just the general definition of an α-iteration)*

Problem 8. *Let κ be a regular uncountable cardinal and $\mathbb{P}$ be a κ-closed poset. Show that $\mathbb{P}$ preserves stationary subsets of κ, i.e. if $S \subset \kappa$ is stationary in the ground model, then S remains stationary in any $\mathbb{P}$-generic extension.*
Hint: Given S, a name $\dot{C}$, and p, such that $p \Vdash$ "$\dot{C}$ is a club subset of κ", show there is a sequence in the ground model $\langle p_\alpha, \gamma_\alpha \mid \alpha < \kappa \rangle$, such that:

- $\langle p_\alpha \mid \alpha < \kappa \rangle$ *is a decreasing sequence below p,*
- $\langle \gamma_\alpha \mid \alpha < \kappa \rangle$ *is a club in κ,*

- *each* $p_\alpha \Vdash \gamma_\alpha \in \dot{C}$.

Then use stationarity of S *in the ground model.*

Problem 9. *Let* $S \subset \omega_1$ *be a stationary set. Define* $\mathbb{P} := \{p \subset S \mid p \text{ is closed and bounded}\}$*, and set* $p \leq q$ *if* p *end extends* q *i.e. for some* α, $p \cap \alpha = q$.

(1) *Show that* $\mathbb{P}$ *is* ω*-distributive, i.e. if* $p \Vdash \dot{f} : \omega \to ON$*, then there is some* $q \leq p$ *and a function* g *in the ground model, such that* $q \Vdash \dot{f} = \check{g}$*. Note that this implies that* $\mathbb{P}$ *adds no countable subsets of* ω_1*, and hence it preserves* ω_1.
(2) *What is the best chain condition for* $\mathbb{P}$*? Justify your answer. Use that and the above to show that* $\mathbb{P}$ *preserves all cardinals.*
(3) *Suppose that* $T := S \setminus \omega_1$ *is also stationary. Let* G *be a* $\mathbb{P}$*-generic filter. Show that in* $V[G]$, T *is nonstationary.*

Remark 2. The above is an example of a forcing that destroys a stationary set, without collapsing cardinals. On the other hand you cannot use forcing to destroy a club set. More precisely, if $V \subset W$ are two models of set theory and $V \models$ "D is club in κ", then $W \models$ "D is club in κ". Note that in the above problem it was important that S was stationary; i.e. you cannot add a new club through a nonstationary set.